

Mathematical modeling and discussions on knowledge for teaching: interactions towards the continuing education of mathematics teachers

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Abstract

In this article, we present our reflections on some of the interactions between Mathematical Modeling and the discussions concerning the professional knowledge of teachers. These interactions, as proposed through the continuing education of early years teachers, show that coordination between these two fields is beneficial to the training of educators, both facilitating the learning of mathematical content and contributing in various ways to the development of skills used to investigate and understand the role mathematics plays in society. Furthermore, the article highlights how these discussions have proved valuable in allowing teachers to understand the importance of possessing specialized content knowledge for teaching mathematics, as posited in the Mathematical Knowledge for Teaching model, developed by Ball, Thames and Phelps (2008).

Keywords: Mathematical Knowledge for Teaching; Teacher Education; Mathematical Modeling; Early Years Education

Interlocuções entre a modelagem e as discussões sobre o conhecimento matemático para o ensino na formação continuada de professores que ensinam matemática

Resumo

Apresentamos, nesse artigo, nossas reflexões acerca de algumas interlocuções entre a Modelagem Matemática e as discussões sobre o conhecimento profissional do professor. As interlocuções propostas, por meio de uma formação continuada de professores que ensinam matemática nos anos iniciais, destacam que a articulação entre essas duas áreas favorece a formação dos professores, promovendo tanto subsídios para a aprendizagem de conteúdos matemáticos como diferentes aportes para o desenvolvimento de habilidades para investigação e compreensão do papel da matemática na sociedade. Destaca-se, ainda, o mérito da percepção de que as discussões possibilitaram, aos professores, o entendimento sobre a importância de um domínio especializado para o ensino de matemática, conforme preconiza o modelo teórico O Conhecimento Matemático para o Ensino, proposto por Ball, Thames e Phelps (2008).

Palavras-chave: Conhecimento Matemático para o Ensino; Formação de Professores; Modelagem Matemática; Anos Iniciais.

Introduction

In this article, we present our reflections on a research study conducted that is at the intersection of two knowledge fields – Teacher Knowledge and Mathematical Modeling¹. We have accepted this intersection because the research was based on a continuing education course for educators who teach mathematics to young students², and proposed from a learning environment with the use of Modeling, from the perspective addressed by Barbosa (2001, 2002).

We sought, therefore, a coordination between a Modeling environment, from the perspective proposed by Barbosa (2001, 2002), and the discussions proposed through the Mathematical Knowledge for Teaching model proposed by Ball

and her colleagues (BALL, HILL and BASS (2005), BALL, THAMES and PHELPS (2008), and BALL, HILL and SHILLING (2008)).

It is worth noting that debates on this topic have been supported by the continuous commotion that has been presented by several studies (SERRAZINA, 2010, 2012; BALL; BASS, 2003; PINO-FAN; GODINO, 2015; PINO-FAN, ASSIS, CASTRO, 2015) that address the professional knowledge of teachers.

Throughout this article, we present the current discussions on these respective areas, as well as the perceived interactions between them and which have contributed to expanding the mathematical knowledge for the education of the teachers who attended the said continuing education course.

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Mathematical Modeling and Teacher Education

Our Modeling literature review³ showed that the Mathematics Education area has a significant number of publications on this subject. Despite this finding, the research performed by Tambarussi and Klüber (2014) showed that *stricto sensu* research discussing teacher education and Modeling specifically is still quite recent.

This conclusion notwithstanding, the Modeling area has been proving its importance as a theoretical and research field. To Souza and Barbosa (2014, p. 32), for example,

the arguments about the implications of using the modeling in an academic and school context for mathematical learning have resulted in its consolidation as a field of research in the area of Mathematics Education.

When looking at studies in the Modeling field, we can also notice that they took different approaches to this topic, which is sometimes presented by multiple points of view and based on different theoretical perspectives (ALMEIDA; VERTUAN, 2011).

Thus, we can conclude that the term "Mathematical Modeling" does not have one single meaning in the context of Mathematics Education. However, what we can in fact observe in the different studies on the topic is that there is a certain degree of consensus among the researchers of this area that Modeling, in general, uses mathematics to understand and solve problem-situations from knowledge areas other than mathematics (BASSANEZI, 2002; BARBOSA, 2001; CALDEIRA 2009).

To Caldeira (2009), Modeling is more than an alternative or teaching methodology, and should be understood as a learning system that makes it possible for people to question mathematical contents, at the same time that it introduces dynamism into its understanding, which must be problematized through a more versatile and critical curriculum that takes into consideration the needs of the time and society.

Caldeira (2009) is against the concept put forward by modern science that divides the curriculum, because, in his view, assembling all the divided parts into a whole is a very difficult task, and this concept causes a fragmented learning as well, with students learning "by parts." Still according to Caldeira (2009), this does not happen

in the Modeling process since knowledge there is not divided, but interconnected and continuous.

By agreeing with these authors, we understand that Modeling can be justified within a curricular proposal that prioritizes school areas as learning environments (BARBOSA, 2002, 2006, 2007), offering plenty of room for the research and analysis of problems that are pertinent to several fields of knowledge or daily activities. Thus, we can observe that Modeling helps the investigation of other knowledge areas through mathematics, since

Modeling is in itself a learning environment in which students are invited to question and/or search, through mathematics, for situations from other knowledge areas. If we analyze Modeling from a sociocritical point of view, the inquiry goes beyond the formulation or understanding of a problem, integrating Mathematical, Modeling and Reflexive knowledge (BARBOSA, 2002, p. 06).

We understand that this perception of the use of Modeling can also be applied to the development of a framework for the continuing education of teachers and the arguments made by these authors enable us to recognize and understand that working with Modeling in teacher education should also mean the development of a work with real, non-mathematical situations in which the use of mathematical concepts and results should be seen as means to discuss and solve problems found in these real situations. This concept is supported by D'Ambrósio's ideas when he states that "modeling is a very rich process of facing situations and results in the effective solution of the real problem rather than simply providing a formal solution to an artificial problem" (D'AMBROSIO, 1986, p. 102).

In our research, we have adopted the Modeling discussions proposed by the cited authors, such as Barbosa (2002, 2006, 2007) and Caldeira (2009,2015), given that we corroborate the idea that the integration among Mathematical, Modeling and Reflexive knowledge comes from the association of the Modeling environment to the problematization and investigation, where the problematization is characterized by the creation of questions and/or problems and the investigation enables, according to Barbosa (2002, p. 7).

the search, selection, organization and manipulation of information and reflection on it [...] In this sense, these questions and investigations can reach the scope of reflexive

knowledge.

In order to foster this much-necessary reflection and involvement of teachers, it is necessary to rethink the way the initial and continuing education of these professionals have developed in relation to the offer of practices that contribute to giving them effective participation, with opinions and reflections with respect to the mathematics teaching and learning process. Barbosa (2001), when referring to the teacher education, suggests that

[...] teacher education involves and is shaped by opinions, questions and/or inquiries. The purpose is to build conditions for reflection on the experiences of teachers and educators (BARBOSA, 2001, p. 55).

The importance of rethinking teacher education and, in particular, their continuing education, is at the heart of the discussion we propose in our research, considering that we agree with the idea that a teacher's performance consists of making decisions in a process by which they build and form their own professional identity.

We understand that one of the characteristics of continuing education of mathematics teachers should be based on the actions that challenge their beliefs and concepts of mathematics itself and how it is taught. Our study, in particular, uses Modeling to instigate these challenges as a way to promote the discussion about teaching work.

Barbosa (2001) proposes a reflection on how to challenge these beliefs and concept teachers have by stating that "once the concepts are formed in the set of experiences, we must use these to unbalance deep-seated concepts" (BARBOSA, 2001, p.5). Thus, the continuing education of teachers should foster their experiences, leading them to reflect on them, that is, the training programs should not lose sight of the practical or professional knowledge of teachers.

These elements discussed so far provide us with support to highlight the importance of discussing the essential knowledge an educator must possess to teach mathematics, since this recognition enables us to understand that the greater the teacher's knowledge, the more conditions they will have to contribute to the teaching and learning

process of mathematics.

Mathematical Knowledge for Teaching

When observing the importance of discussing the knowledge that can be considered essential for an educator to teach mathematics, we are confronted with the primordially of addressing studies on the professional knowledge of the teacher.

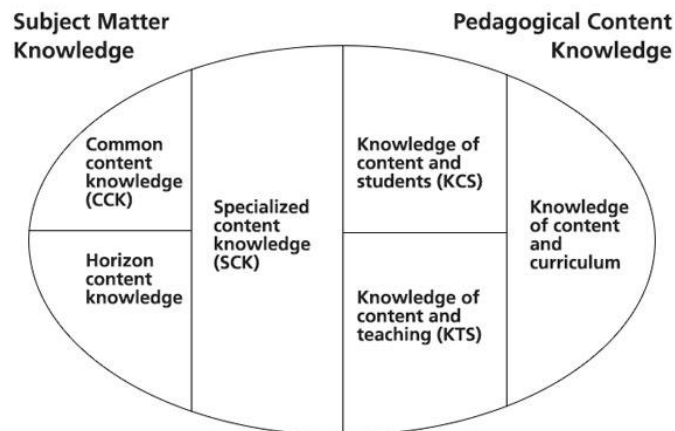
With this in mind, as we progressed in the readings of the authors who discuss this subject, we came across a great number of theoretical perspectives that prioritize this topic in their discussions. Generally, this knowledge is discussed as a combination of the training and experience teachers have and use in the development of their teaching practice, which is built through the teacher's own development, whether as an individual or as a professional, and which continues throughout their teaching career.

These readings have also enabled us to observe that there are several categories of professional knowledge that seek to identify and discuss the skills a teacher must have to teach properly. As examples, we can cite the discussions proposed by Ball, Thames and Phelps (2008), Godino (2009), Pino-Fan, Assis and Castro (2015), Pino-Fan and Godino (2015), Shulman (1986, 1987) and Garcia (1999), among others.

Of these researchers and over the last few years, the papers published by Ball (2000), Ball, Hill and Bass (2005), Hill, Rowan and Ball (2005), Ball, Thames and Phelps (2008), and Ball, Hill and Shilling (2008) have investigated and discussed which knowledge a teacher should have so they can teach mathematics.

From the systematization of her studies, Ball and her colleagues presented a Mathematical Knowledge for Teaching (MKT) model. The following Figure is presented by Ball, Thames and Phelps (2008) to illustrate the proposed model of the Domains of Mathematical Knowledge for Teaching, where each one of the six divisions in the figure is a proposed element of this knowledge. By Mathematical Knowledge for Teaching, the authors refer to the mathematical knowledge necessary for someone to perform the work of teaching mathematics.

Domains of Mathematical Knowledge for Teaching⁴



However, the authors emphasize that this map is not definitive, since the discussions on these domains are in continuous construction.

These authors' discussions, based on the map presented in the previous figure, were decisive for us to use this theoretical reference to analyze the actions of a group of teachers during the continuing education meetings that were focused on the Modeling assumptions previously presented.

During the course meetings, we focused on investigating how Specialized Content Knowledge and the Knowledge of Content and Teaching emerged within the Modeling environment. Thus, we explored the Modeling environment as a possibility to offer us the conditions to understand, in greater detail, how and which elements of mathematical content and practice are or should be used by teachers, when they choose their teaching lines.

For a better understanding of these two sub-domains proposed by the theoretical model presented by Ball, Hill and Bass (2005), we describe them in greater depth.

Specialized Content Knowledge

Based on the description we presented of the sub-domains discussed by the authors, we can think of the following situation, as an example and differentiation between a situation that involves Common Content Knowledge and Specialized Content Knowledge: by recognizing a wrong answer to a given calculation, this ability of identifying an error can be seen as a common content knowledge, while classifying the nature of the error, especially an unknown error, usually requires quick numerical reasoning, attention to patterns, and flexible thinking on determined

meanings, thus this situation is characteristic of Specialized Content Knowledge.

To Ball and Bass (2003), teachers need to know how to justify, analyze errors, generalize and propose definitions. To do so, they need to have the knowledge of ideas and procedures, as well as the skills to represent and explain them appropriately to the students. Ball, Hill and Bass (2005) use a simple situation and that can be seen as a characteristic of Specialized Content Knowledge, by presenting a multiplication of integral numbers. For the authors, an aspect of this knowledge is being able to use a reliable algorithm to calculate the answer. They presented the following multiplication problem: 33×25 .

The authors point out that most people should remember how to proceed, or the algorithms they have learned, which results in the following: $35 \times 25 = 87$

However, what these authors highlight is that being able to perform a multiplication correctly is essential knowledge to teach multiplication to the students, but that alone is not enough. The authors classify this knowledge as common content knowledge, that is, a professional from other areas may be able to solve this problem.

Based on this argument, Ball, Hill and Bass (2005) suggest that teachers, when searching for patterns in the students' errors, or when analyzing whether a non-standardized approach to teaching would work, usually need to perform a kind of mathematical work that other professionals cannot, that is, it involves a kind of mathematical work that is not necessary in situations other than teaching.

Thus, we can observe that many of the daily tasks of teaching are characteristic to this special work that needs to be performed by teachers, who should carry out these tasks regularly. However, as

pointed out by this discussion, the accomplishment of these tasks requires mathematical understanding and thinking that are particular to teaching and which characterize specialized content knowledge, that is, the mathematical demands of teaching require the specialized knowledge of mathematics that is not necessary in other situations.

Mizukami (2004) also emphasizes the importance of this type of knowledge for teaching, and his ideas corroborate the discussions of Ball, Hill and Bass (2005), wherein although this knowledge is necessary and relevant to teaching, possessing it alone does not guarantee that it will be taught and learned successfully, that is, this knowledge is necessary, but not enough.

Knowledge of Content and Teaching

The second sub-domain of Mathematical Knowledge for Teaching that we propose to discuss throughout this article refers to the Knowledge of Content and Teaching. This type of knowledge should enable teachers to reflect, for example, on questions such as: (i) *Is this activity important to my students?*; (ii) *Which patterns and nuances would this activity lead my students to understand?*; (iii) *Is this activity worth in terms of what the students can learn from it?* These questions can be seen as guides so that teachers can decide how interesting and important a certain activity can be to students, in addition to being able to evaluate how difficult this activity is (BALL, 2000).

Ball and Bass (2003) highlight that the example of these questions for the preparation and analysis of a single mathematical activity reveals how much the essential teaching tasks involve putting significant mathematical thinking into practice, especially when we know that this analysis represents only one fraction of the work a teacher needs to do to make the use of this problem productive with the students.

These authors point out that

teaching requires, therefore, a special type of sensitivity to the need of accuracy in mathematics. Accuracy requires that the language and ideas are meticulously specified to solve mathematical problems, so that they are not unnecessarily hindered by ambiguities in meaning and interpretation. However, the need for accuracy is relative and depends on the context and use (BALL; BASS, 2003, p. 8) [our translation].

At the heart of the discussions, we can say that there should be a concern with the extension

and nature of the mathematical knowledge required for teaching. Ball, Hill and Bass (2005) point out that only a few studies have addressed what should be the appropriate mathematical "curriculum" for the teacher education, so that that the teachers can learn the appropriate mathematics and contribute to the learning of the students.

In the context of this discussion, the authors present the map based on the practice they call Mathematical Knowledge for Teaching, which we described previously. Therefore, this knowledge is seen by the authors as a kind of professional mathematics knowledge, different from the one required by other professionals that also use the mathematics on a daily basis, such as Engineering, Physics, Accounting, among others, as already exemplified. We can say that this knowledge is only understood through initial or continuing education of teachers, because this is when the discussions on these type knowledges can be carried out.

According to these authors, one fundamental point teachers find it difficult to learn to teach, despite its centrality, is in knowledge of mathematical content, which, in most cases, is not something taught efficiently in teacher education. Therefore, although some teachers have a significant degree of knowledge of mathematical content, this is often not enough to help them to listen to the students, choose good activities, or help the students to learn.

The discussions observed through the presentation of these two sub-domains refer us to Ball's (2000) statement that points out the need to create opportunities for teachers to learn mathematical content in a way that enables them not only to understand the content but also to have the opportunity to learn how to use what they know in a variety of practical contexts. Thus, the author emphasizes that understanding what the teachers need to know, how they need to know and help them to learn how to use it, since they are the factors that support the problem of the preparation of content by the teachers in the practice, could help to fill the gaps that sometimes prevent progress in teacher education.

It is a fact that every teacher needs to have a good grasp of the content that they must teach, so that they can understand the knowledge building process of their students and contribute to their learning by presenting didactic situations. The studies produced by Ball (1991, 2004) highlight the fact that the academic success of students does not rest solely on the teacher's specialized knowledge of mathematics – it is also contingent upon the teacher's ability to establish connections between this type of knowledge and the knowledge they have

of their students, taking their learning processes into consideration.

Reflections on the interaction between the Mathematical Modeling and discussions on Mathematical Knowledge for Teaching

In order to present our reflections on this interaction, we chose to build a dynamic that is divided into two sections: in the first section, "*Describing and Analyzing an Education Meeting*", we describe one of the continuing education course meetings and do a preliminary analysis of the data we extracted from it, based on our theoretical reference. In the second section, "*Analyzing the Intentionality of the Meeting*", we summarize the meeting.

Describing and Analyzing an Education Meeting

At a previous continuing education meeting, we asked the teachers to identify, within the campus of the university where the meetings were being held, a physical space where a sport court could be built. Thus, during this meeting we are describing, the instructor referred back to some questions to determine the "floor area" needed for this construction.

As the teachers needed to determine the "floor area" of the land, the instructor asked them: "When we think of square meter, what is this unit of measurement related to? Teacher Simone⁵ answered that the area is related to the perimeter, and teacher Milena said: "the area is related to the sides." Teacher Tabatha added: "one is the sum of the sides and the other is the area. But I can't remember which is which. I think that the area is the base times the height. This is how I explain it to my students.

The teachers' answers show some relationships, doubts and misconceptions regarding the mathematical concepts and aspects related to teaching and learning of mathematics that are very limited and marked by some misunderstandings. In some respects, we noted some gaps in the teachers' knowledge about the contents they teach and how they can be learned.

The intentionality of the dialogue focused on developing the understanding of the relations between perimeter, area and sides through a process in which the teachers, when their conceptions regarding these relations conflicted, could build new knowledge and validate their arguments.

The apparent confusion between the concepts of perimeter and area shown by some teachers is also very common among students. Pires

(2012), when discussing the problems related to these concepts, present some theories to explain these errors and highlights that one of them is that "perimeters and areas are traditionally taught by almost immediately presenting formulas to be applied to problem-situations" (PIRES, 2012, p. 240).

Pires' (2012) statements reaffirm our understanding that in order to propose and carry out activities with students with the aim of minimizing problems such as those we pointed out, teachers must have both specialized content knowledge and knowledge of content and teaching, as stressed by Ball, Thames and Phelps (2008).

The instructor sought, whenever possible, to encourage the teachers to explain their mathematical ideas, beliefs and conceptions explicitly, and refrained from correcting them immediately even when they used a misconception. In this Modeling context, teachers would ideally play an active role in building their own knowledge and, thus, learn mathematics as an achievement, that is, mathematics as a process and not as a rigid and finished science. An example of this situation can be seen in the following dialogue:

Instructor: "Simone, with the side measurements you found, what would be the total area that we can consider, before thinking of the distance from the stream?"

Teacher Simone: "I found two hundred and sixty-six."

Instructor: "Two hundred and sixty-six what?"

Teacher Simone: "Two hundred and sixty-six square meters."

Instructor: "And how did you find this number?"

Teacher Simone: "Ah, I added up all the sides."

This dialogue led to some discussions within the group. Some teachers agreed with Simone, while some did not. Teacher Milena brought something very interesting to this discussion. At first, she thought that teacher Simone was right. But it was evident that she had doubts, and kept reflecting on the issue. Then, after some time, she said: "I think it's wrong [...] because I bought a land lot and I know it has eight hundred square meters, twenty meters in the front and forty meters deep. It is also a rectangle. So you have to multiply. If I add it all up, my result is not eight hundred square meters [...] That's it, you have to multiply, don't you? I think so. It has to be. That's the only way it will work out."

We can see that Teacher Milena used a real example, which is part of her personal context, to

argue about the validity of a mathematical relation. Her argument was not based on mathematical properties or formulas, but rather on common content knowledge that is supported by her experience, because there is no going against the fact that her land lot had those measures. As the discussion involved two possibilities - add up the length of all sides or multiply the width by the depth, the teacher concluded that the correct option was to multiply the sides of the land lot.

To Pino-Fan and Godino (2015), situations such as this demonstrate that problems can indeed be solved with common content knowledge. However, this situation also showed that the teachers have difficulty when they face items that aim to explore other dimensions of their knowledge, such as the ones related to specialized content knowledge.

When the teachers were measuring the land area, they also had to conjure up enough common content knowledge to record the information and survey the sides, although without the accuracy necessary to delimit them, and, through this discussion, they reached a conclusion on the relation between the land sides and the area calculation.

However, what they did and what they said during both the measurement process and these discussions showed that they required specialized knowledge content, given that, in addition to possessing mathematics knowledge that would enable them to solve problems by using common content knowledge, the teachers required mathematical knowledge that is adequate to "teaching [...], providing several justifications and arguments, and identifying mathematical knowledge in use during the process of solving a mathematical activity" (PINO-FAN; ASSIS; CASTRO, 2015, p. 1434) [our translation].

At a different moment of this meeting, the instructor asked the teachers: "Can we find the object faces in both plane and spatial figures?" No teacher answered the question. Then, teacher Milena asked the instructor to define face. However, the instructor did not want to do it right at that time so they could talk more about it. Teacher Milena finally said: "I think so. All figures have faces." She then asked if she could go to the blackboard to show her reasoning.

She drew a quadrangular figure on the board and said: "Look, the square has a face. The square face. And if you have a cube, you have multiple faces." At this moment, all the other teachers followed teacher Milena's explanation and most of them agreed with her: "That's right," said teacher Tabatha. "Squares and cubes have faces," said teacher Evelyn.

The instructor, then, intervened and asked them: "So if we think of two-dimensional figures, what characterizes a two-dimensional figure?"

Teacher Marjory: I've never heard of it.

Teacher Milena: It is a plane figure that has two dimensions.

Instructor: What are the dimensions?

Teacher Milena: Height and length.

Despite having answered that two-dimensional figures are the ones that have height and length, teacher Milena confessed that she has difficulty explaining what is the height, width, length and depth of an object to her students.

From the discussion that arose about these elements, the instructor initiated a discussion, based on a problematization, of the classifications of the mathematical objects in relation to their dimension, always asking questions to the teachers as a way of involving them in the discussion and having them share their ideas and conceptions of the addressed subjects.

In order to investigate how the teachers would explain their conceptions, the instructor asked them: "What about the three-dimensional geometric figures? Can we grab three-dimensional geometric figures?" At that moment, the concern expressed by the instructor was confirmed as most of the teachers answered affirmatively. As a way of illustrating their reaction, part of the dialogue between the instructor and the teachers is described below:

Instructor: Take a milk box; which mathematical object can we associate it with?

Teachers: Parallelepipeds.

Instructor: But, isn't the parallelepiped a geometric figure?

Teachers: Yes.

Instructor: But geometric figures are types of mathematical objects. They are abstract objects, also called ideal objects, but which are only part of our mind. They are not tangible objects. We cannot grab or touch them.

At this moment the teachers felt very uncomfortable with this statement, and teacher Milena commented: "I'm lost. I have always worked with this box saying it was a parallelepiped." The instructor then replied: The milk box is a representation of the parallelepiped. This box represents a three-dimensional geometrical figure, which is the parallelepiped.

The discussion on this subject was very productive and enlightening, because it contributed to make the teachers reflect on their practices. Not only did they realize that they required knowledge of what they were discussing, they recognized the importance of using coherent terms and language to

present content to their students.

Menezes (2000), when giving a lecture on "Mathematics, Language and Communication," emphasized that

teachers' practices have a strong language component. These practices are often impregnated with the teachers' views and values, among others, as to where do language and communication stand in the teaching and learning of mathematics. The language used in mathematics classes is also influenced by other factors than the teachers' conceptions, such as student background, socio-cultural level and teacher education (MENEZES, 2000).

This was a very important observation, as language should be one of teachers' concerns, since inadequate language can induce students to errors or misconceptions.

We believe that this discussion helped the teachers to assimilate the implicit mathematical concepts existing in the investigated situation.

Analyzing the Intentionality of the Meeting

The instructor's intentionality throughout the meetings always was to uncover the underlying discussions of the specific contents of mathematics, as a way of raising, problematizing and establishing a relation between these discussions, specialized content knowledge and knowledge of content and teaching.

Thus, in addition to the specific discussions on mathematical contents, which were important and necessary during the meeting, we emphasize that the instructor did what he did to bring these underlying elements to light for the purposes of problematization.

We can say that the focus of this meeting was to discuss mathematical objects and their representations and this discussion contributed to the teachers' understanding of the importance of this knowledge for their teaching practices.

The instructor, throughout the meetings, and more incisively in this particular meeting, used pedagogical questions as a way to involve the teachers in the proposed discussions. 2001 p. 57) refers to pedagogical questions as a "strategy for the development of the capacity of reflection." To this author, pedagogical questions "must have a formative intentionality" (p. 57), and the instructor used them for this precise purpose.

We understand that problematization involving mathematical objects and their representations led to a very productive discussion

due to the importance of addressing these topics in teacher education, both initial and continuing, because of its importance to the teacher's practice in the classroom.

We would also like to emphasize that the teachers said that the discussion involving the concept of dimension and the classification of mathematical objects according to their dimension was totally new for them. We observed that this discussion helped the teachers to understand the ideas related to the objects and their representations, since the use of representations proved to be essential when working with one-dimensional or no-dimension objects to refer to the mathematical objects belonging to these dimensions.

Conclusions

Providing continuing education to teachers using Modeling as a learning environment allowed us to see how they used their mathematical knowledge to perform the proposed activities, which enabled us to collect some evidence of their mathematical knowledge for teaching. We also point out that, through this learning environment, Modeling contributed to the very development of mathematical knowledge for teaching, whether through the use of common mathematical content knowledge or through the transformation of other types of knowledge related to this common knowledge into specialized mathematical content knowledge. Thus, we can say that the work with Mathematical Modeling as a learning environment proved to be a privileged field for this development.

To Ball (2000), it is necessary to create opportunities for teachers to not only learn the mathematical content, but also how to use it in different practical contexts. This way, Ball emphasizes that understanding what the teachers need to know, how they need to know, and helping them to learn how to use this knowledge, since these are the factors that support the problem of the preparation of content by teachers, could help fill the gaps that sometimes prevent progress in teacher education.

We can also conclude that the proposed activity involving Modeling allowed the teachers, as affirmed by Ball and Bass (2003), to realize that activities of this nature improve their ability to connect contents, whether they belong to a particular mathematical domain or extend across the different education levels, contributing to the students' understanding of the relation between the mathematical ideas and concepts that they are learning, and allowing them to establish their own connections between the different areas of

mathematics, such as geometry and arithmetic, for example.

The Modeling environment has proved to be the driving force to promote an education method that has at its core the ideas proposed by Ball, Thames and Phelps (2008) that the discussions on Mathematical Knowledge for Teaching have to problematize teaching situations, that is, such method must prioritize the understanding of the tasks involved in teaching, as well as their mathematical requirements.

Notes

- 1 We will use the term Modeling to refer to Mathematical Modeling in Mathematics Education.
- 2 We use the expression "educators who teach mathematics to young children" to refer to the educators who teach early years of Primary School (6-10 years old).
- 3 We will use the term Modeling to refer to the Mathematical Modeling in Mathematics Education so as to avoid repetition.
- 4 Model of the Domains of Mathematical Knowledge for Teaching map. Source: Ball, Thames and Phelps (2008).
- 5 The teachers' names were changed in order to ensure anonymity.

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